SECTION 7.8 EXERCISES

Review Questions

1. What is the order of $y''(t) + 9y(t) = 10$?

2. Is $y''(t) + 9y(t) = 10$ linear or nonlinear?

3. How many arbitrary constants appear in the general solution of $y''(t) + 9y(t) = 10$?

4. If the general solution of a differential equation is $y(t) = Ce^{-3t} + 10$, what is the solution that satisfies the initial condition $y(0) = 5$?

5. What is a separable first-order differential equation?

6. Is the equation $t^2 y'(t) = (t + 4)/y^2$ separable?

7. Explain how to solve a separable differential equation of the form $g(y) y'(t) = h(t)$.

8. Explain how to sketch the direction field of the equation $y'(t) = F(t, y)$, where F is given.

Basic Skills

9–12. Warm-up initial value problems *Solve the following problems.*

9. $y'(t) = 3t^2 - 4t + 10, \ y(0) = 20$

10. $\dfrac{dy}{dt} = 8e^{-4t} + 1, \ y(0) = 5$

11. $y'(t) = (2t^2 + 4)/t, \ y(1) = 2$

12. $\dfrac{dy}{dx} = 3\cos 2x + 2\sin 3x, \ y(\pi/2) = 8$

13–16. First-order linear equations *Find the general solution of the following equations.*

13. $y'(t) = 3y - 4$

14. $\dfrac{dy}{dx} = -y + 2$

15. $y'(x) = -2y - 4$

16. $\dfrac{dy}{dt} = 2y + 6$

17–20. Initial value problems *Solve the following problems.*

17. $y'(t) = 3y - 6, \ y(0) = 9$

18. $\dfrac{dy}{dx} = -y + 2, \ y(0) = -2$

19. $y'(t) = -2y - 4, \ y(0) = 0$

20. $\dfrac{du}{dx} = 2u + 6, \ u(1) = 6$

21. Intravenous drug dosing The amount of drug in the blood of a patient (in mg) due to an intravenous line is governed by the initial value problem

$$y'(t) = -0.02y + 3, \quad y(0) = 0 \qquad \text{for } t \geq 0$$

where t is measured in hours.

a. Find and graph the solution of the initial value problem.
b. What is the steady-state level of the drug?
c. When does the drug level reach 90% of the steady-state value?

22. Fish harvesting A fish hatchery has 500 fish at time $t = 0$, when harvesting begins at a rate of b fish/yr, where $b > 0$. The fish population is modeled by the initial value problem

$$y'(t) = 0.1y - b, \quad y(0) = 500 \qquad \text{for } t \geq 0$$

where t is measured in years.

a. Find the fish population for $t \geq 0$ in terms of the harvesting rate b.
b. Graph the solution in the case that $b = 40$ fish/yr. Describe the solution.
c. Graph the solution in the case that $b = 60$ fish/yr. Describe the solution.

23–26. Separable differential equations *Find the general solution of the following equations.*

23. $\dfrac{dy}{dt} = \dfrac{3t^2}{y}$

24. $\dfrac{dy}{dx} = y(x^2 + 1)$, where $y > 0$

25. $y'(t) = e^{y/2} \sin t$

26. $x^2 \dfrac{dw}{dx} = \sqrt{w}(3x + 1)$

27–32. Separable differential equations *Determine whether the following equations are separable. If so, solve the given initial value problem.*

27. $\dfrac{dy}{dt} = ty + 2, \ y(1) = 2$

28. $y'(t) = y(4t^3 + 1), \ y(0) = 4$

29. $y'(t) = \dfrac{e^t}{2y}, \ y(\ln 2) = 1$

30. $(\sec x)\, y'(x) = y^3, \ y(0) = 3$

31. $\dfrac{dy}{dx} = e^{x-y}, \ y(0) = \ln 3$

32. $y'(t) = 2e^{3y-t}, \ y(0) = 0$

33. Logistic equation for a population A community of hares on an island has a population of 50 when observations begin at $t = 0$. The population for $t \geq 0$ is modeled by the initial value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{200}\right), \quad P(0) = 50$$

a. Find and graph the solution of the initial value problem.
b. What is the steady-state population?

34. Logistic equation for an epidemic When an infected person is introduced into a closed and otherwise healthy community, the number of people who become infected with the disease (in the absence of any intervention) may be modeled by the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right), \quad P(0) = P_0$$

where k is a positive infection rate, A is the number of people in the community, and P_0 is the number of infected people at $t = 0$. The model assumes no recovery or intervention.

a. Find the solution of the initial value problem in terms of k, A, and P_0.

b. Graph the solution in the case that $k = 0.025$, $A = 300$, and $P_0 = 1$.
c. For fixed values of k and A, describe the long-term behavior of the solutions for any P_0 with $0 < P_0 < A$.

35–36. Direction fields *A differential equation and its direction field are given. Sketch a graph of the solution that results with each initial condition.*

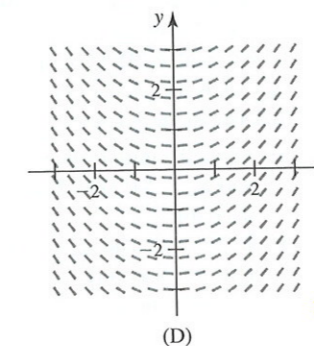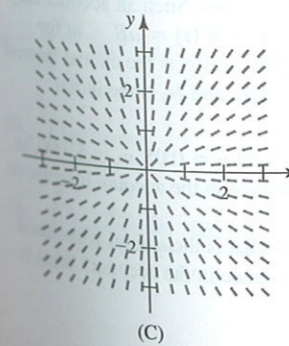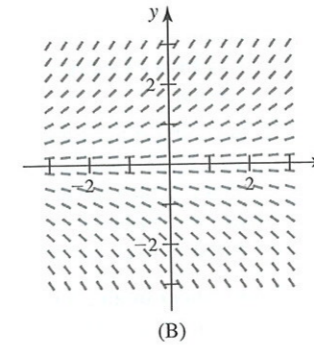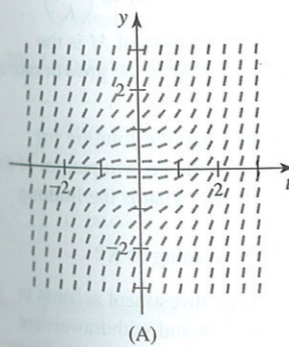35. $y'(t) = \dfrac{t^2}{y^2 + 1}$, $\ y(0) = -2$ and $\ y(-2) = 0$
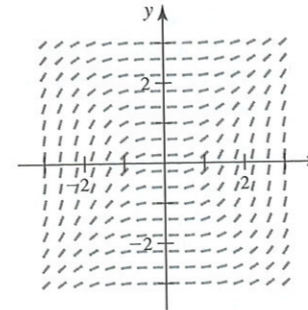
36. $y'(t) = \dfrac{\sin t}{y}$, $\ y(-2) = -2$ and $\ y(-2) = 2$
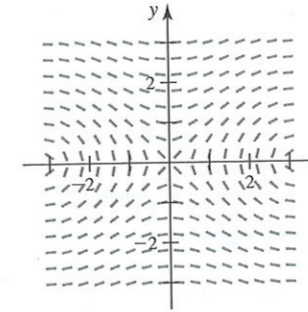
37. **Matching direction fields** Match equations (a)–(d) with the direction fields A–D.

(a) $y'(t) = t/2$ (b) $y'(t) = y/2$
(c) $y'(t) = (t^2 + y^2)/2$ (d) $y'(t) = y/t$

(A)

(B)

(C)

(D)

38–40. Sketching direction fields *Use the window $[-2, 2] \times [-2, 2]$ to sketch a direction field for the following equations. Then sketch the solution curve that corresponds to the given initial condition.*

38. $y'(t) = y - 3, \ y(0) = 1$

39. $y'(x) = \sin x, \ y(-2) = 2$

40. $y'(t) = \sin y, \ y(-2) = \frac{1}{2}$

Further Explorations

41. **Explain why or why not** Determine whether the following statements are true and give an explanation or counter-example.

a. The general solution of $y'(t) = 20y$ is $y = e^{20t}$.
b. The functions $y = 2e^{-2t}$ and $y = 10e^{-2t}$ do not both satisfy the differential equation $y' + 2y = 0$.
c. The equation $y'(t) = ty + 2y + 2t + 4$ is not separable.
d. A solution of $y'(t) = 2\sqrt{y}$ is $y = (t + 1)^2$.

42–47. Equilibrium solutions *A differential equation of the form $y'(t) = F(y)$ is said to be* **autonomous** *(the function F depends only on y). The constant function $y = y_0$ is an equilibrium solution of the equation provided $F(y_0) = 0$ (because then $y'(t) = 0$, and the solution remains constant for all t). Note that equilibrium solutions correspond to horizontal line segments in the direction field. Note also that for autonomous equations, the direction field is independent of t. Consider the following equations.*

a. Find all equilibrium solutions.
b. Sketch the direction field on either side of the equilibrium solutions for $t \geq 0$.
c. Sketch the solution curve that corresponds to the initial condition $y(0) = 1$.

42. $y'(t) = 2y + 4$

43. $y'(t) = y^2$

44. $y'(t) = y(2 - y)$

45. $y'(t) = y(y - 3)$

46. $y'(t) = \sin y$

47. $y'(t) = y(y - 3)(y + 2)$

48–51. Solving initial value problems *Solve the following problems using the method of your choice.*

48. $u'(t) = 4u - 2, \ u(0) = 4$

49. $\dfrac{dp}{dt} = \dfrac{p + 1}{t^2}, \ p(1) = 3$

50. $\dfrac{dz}{dx} = \dfrac{z^2}{1 + x^2}, \ z(0) = \dfrac{1}{6}$

51. $w'(t) = 2t\cos^2 w, \ w(0) = \pi/4$

52. **Optimal harvesting rate** Let $y(t)$ be the population of a species that is being harvested. Consider the harvesting model $y'(t) = 0.008y - h, y(0) = y_0$, where $h > 0$ is the annual harvesting rate and y_0 is the initial population of the species.

a. If $y_0 = 2000$, what harvesting rate should be used to maintain a constant population of $y = 2000$ for $t \geq 0$?
b. If the harvesting rate is $h = 200$/year, what initial population ensures a constant population for $t \geq 0$?

Applications

T 53. Logistic equation for spread of rumors Sociologists model the spread of rumors using logistic equations. The key assumption is that at any given time a fraction y of the population, where $0 \le y \le 1$, knows the rumor, while the remaining fraction $1 - y$ does not. Furthermore, the rumor spreads by interactions between those who know the rumor and those who do not. The number of such interactions is proportional to $y(1 - y)$. Therefore, the equation that describes the spread of the rumor is $y'(t) = ky(1 - y)$, where k is a positive real number. The fraction of people who initially know the rumor is $y(0) = y_0$, where $0 < y_0 < 1$.

a. Solve this initial value problem and give the solution in terms of k and y_0.

b. Assume $k = 0.3$ weeks^{-1} and graph the solution for $y_0 = 0.1$ and $y_0 = 0.7$.

c. Describe and interpret the long-term behavior of the rumor function for any $0 < y_0 < 1$.

T 54. Free fall An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the object falls). By Newton's second law (mass × acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

$$\underbrace{m}_{\text{mass}} \cdot \underbrace{v'(t)}_{\text{acceleration}} = \underbrace{mg + f(v)}_{\substack{\text{external} \\ \text{forces}}},$$

where f is a function that models the resistance and the positive direction is downward. One common assumption (often used for motion in air) is that $f(v) = -kv^2$, where $k > 0$ is a drag coefficient.

a. Show that the equation can be written in the form $v'(t) = g - av^2$, where $a = k/m$.

b. For what (positive) value of v is $v'(t) = 0$? (This equilibrium solution is called the **terminal velocity**.)

c. Find the solution of this separable equation assuming $v(0) = 0$ and $0 < v(t)^2 < g/a$, for $t \ge 0$.

d. Graph the solution found in part (c) with $g = 9.8$ m/s², $m = 1$ kg, and $k = 0.1$ kg/m, and verify that the terminal velocity agrees with the value found in part (b).

T 55. Free fall Using the background given in Exercise 54, assume the resistance is given by $f(v) = -Rv$, where $R > 0$ is a drag coefficient (an assumption often made for a heavy medium such as water or oil).

a. Show that the equation can be written in the form $v'(t) = g - bv$, where $b = R/m$.

b. For what (positive) value of v is $v'(t) = 0$? (This equilibrium solution is called the **terminal velocity**.)

c. Find the solution of this separable equation assuming $v(0) = 0$ and $0 < v < g/b$.

d. Graph the solution found in part (c) with $g = 9.8$ m/s², $m = 1$ kg, and $R = 0.1$ kg/s, and verify that the terminal velocity agrees with the value found in part (b).

T 56. Torricelli's Law An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli's Law (see figure). If $h(t)$ is the depth of water in the tank for $t \ge 0$, then Torricelli's Law implies $h'(t) = -2k\sqrt{h}$, where $k > 0$ is a constant that includes the acceleration due to gravity.
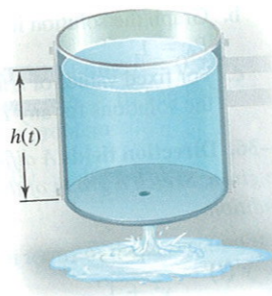
the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is $h(0) = H$.

a. Find the general solution of the equation.

b. Find the solution in the case that $k = 0.1$ and $H = 0.5$ m.

c. In general, how long does it take for the tank to drain in terms of k and H?

T 57. Chemical rate equations The reaction of chemical compounds can often be modeled by differential equations. Let $y(t)$ be the concentration of a substance in a reaction for $t \ge 0$ (typical units of concentration are moles/L). The change in the concentration of the substance, y are moles/L). The change in the concentration of the substance, under appropriate conditions, is $\dfrac{dy}{dt} = -ky^n$, where $k > 0$ is a rate constant and the positive integer n is the order of the reaction.

a. Show that for a first-order reaction ($n = 1$), the concentration obeys an exponential decay law.

b. Solve the initial value problem for a second-order reaction ($n = 2$) assuming $y(0) = y_0$.

c. Graph and compare the concentration for a first-order and second-order reaction with $k = 0.1$ and $y_0 = 1$.

T 58. Tumor growth The growth of cancer tumors may be modeled by the Gompertz growth equation. Let $M(t)$ be the mass of the tumor for $t \ge 0$. The relevant initial value problem is

$$\frac{dM}{dt} = -aM \ln\left(\frac{M}{K}\right), \qquad M(0) = M_0,$$

where a and K are positive constants and $0 < M_0 < K$.

a. Graph the growth rate function $R(M) = -aM \ln\left(\dfrac{M}{K}\right)$ assuming $a = 1$ and $K = 4$. For what values of M is the growth rate positive? For what value of M is the growth rate a maximum?

b. Solve the initial value problem and graph the solution for $a = 1$, $K = 4$, and $M_0 = 1$. Describe the growth pattern of the tumor. Is the growth unbounded? If not, what is the limiting size of the tumor?

c. In the general equation, what is the meaning of K?

T 59. Endowment model An endowment is an investment account in which the balance ideally remains constant and withdrawals are made on the interest earned by the account. Such an account may be modeled by the initial value problem $B'(t) = aB - m$ for $t \ge 0$, with $B(0) = B_0$. The constant a reflects the annual interest rate, m is the annual rate of withdrawal, and B_0 is the initial balance in the account.

a. Solve the initial value problem with $a = 0.05$, $m = \$1000/\text{yr}$, and $B_0 = \$15,000$. Does the balance in the account increase or decrease?

b. If $a = 0.05$ and $B_0 = \$50,000$, what is the annual withdrawal rate m that ensures a constant balance in the account? What is the constant balance?

Additional Exercises

60. Solution of the logistic equation Consider the solution of the logistic equation in Example 4.

a. From the general solution $\ln\left|\dfrac{P}{300 - P}\right| = 0.1t + C$, show that the initial condition $P(0) = 50$ implies that $C = \ln\frac{1}{5}$.

b. Solve for P and show that $P(t) = \dfrac{300}{1 + 5e^{-0.1t}}$.

61. Direction field analysis Consider the general first-order initial value problem $y'(t) = ay + b$, $y(0) = y_0$, for $t \ge 0$, where a, b, and y_0 are real numbers.

a. Explain why $y = -b/a$ is an equilibrium solution and corresponds to a horizontal line in the direction field.

b. Draw a representative direction field in the case that $a > 0$. Show that if $y_0 > -b/a$, then the solution increases for $t \ge 0$, and if $y_0 < -b/a$, then the solution decreases for $t \ge 0$.

c. Draw a representative direction field in the case that $a < 0$. Show that if $y_0 > -b/a$, then the solution decreases for $t \ge 0$, and if $y_0 < -b/a$, then the solution increases for $t \ge 0$.

QUICK CHECK ANSWERS

1. The equation is first order and linear. **3.** The solution is $y(t) = 16e^{3t} - 2$. **4.** $y^3 y'(t) = t^2 + 1$ **5. a.** Solution increases for $t \ge -1$. **b.** Solution decreases for $t \ge -2$. ◄

CHAPTER 7   REVIEW EXERCISES

1. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The integral $\int x^2 e^{2x}\, dx$ can be evaluated analytically using integration by parts.

b. To evaluate the integral $\displaystyle\int \frac{dx}{\sqrt{x^2 - 100}}$ analytically, it is best to use partial fractions.

c. One computer algebra system produces $\int 2 \sin x \cos x\, dx = \sin^2 x$. Another computer algebra system produces $\int 2 \sin x \cos x\, dx = -\cos^2 x$. One computer algebra system is wrong (apart from a missing constant of integration).

2–19. Integrals *Evaluate the following integrals analytically.*

2. $\displaystyle\int x^2 \cos x\, dx$

3. $\displaystyle\int e^x \sin x\, dx$

4. $\displaystyle\int_1^e x^2 \ln x\, dx$

5. $\displaystyle\int \cos^2 4\theta\, d\theta$

6. $\displaystyle\int \sin 3x \cos^6 3x\, dx$

7. $\displaystyle\int \sec^5 z \tan z\, dz$

8. $\displaystyle\int_0^{\pi/2} \cos^4 x\, dx$

9. $\displaystyle\int_0^{\pi/6} \sin^5 \theta\, d\theta$

10. $\displaystyle\int \tan^4 u\, du$

11. $\displaystyle\int \frac{dx}{\sqrt{4 - x^2}}$

12. $\displaystyle\int \frac{dx}{\sqrt{9x^2 - 25}}$ for $x > \dfrac{5}{3}$

13. $\displaystyle\int \frac{dy}{y^2\sqrt{9 - y^2}}$

14. $\displaystyle\int_0^{\sqrt{3}/2} \frac{x^2}{(1 - x^2)^{3/2}}\, dx$

15. $\displaystyle\int_0^{\sqrt{3}/2} \frac{4}{9 + 4x^2}\, dx$

16. $\displaystyle\int \frac{(1 - u^2)^{5/2}}{u^8}\, du$

17. $\displaystyle\int \frac{dx}{x^2 - 2x - 15}$

18. $\displaystyle\int \frac{dx}{x^3 - 2x^2}$

19. $\displaystyle\int_0^1 \frac{dy}{(y + 1)(y^2 + 1)}$

20–22. Table of integrals *Use a table of integrals to evaluate the following integrals.*

20. $\displaystyle\int x(2x + 3)^5\, dx$

21. $\displaystyle\int \frac{dx}{x\sqrt{4x - 6}}$

22. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{1 + \sin 2\theta}$

T 23–24. Approximations *Use a computer algebra system to approximate the value of the following integrals.*

23. $\displaystyle\int_{-1}^1 e^{-2x^2}\, dx$

24. $\displaystyle\int_1^{\sqrt{e}} x^3 (\ln x)^3\, dx$

T 25. Numerical integration methods Let $I = \int_0^3 x^2\, dx = 9$ and consider the Trapezoid Rule $T(n)$ and the Midpoint Rule $M(n)$ approximations to I.

a. Compute $T(6)$ and $M(6)$.

b. Compute $T(12)$ and $M(12)$.